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THE HAM SANDWICH THEOREM REVISITED

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ABSTRACT

This paper continues the search, started in [10], for relatives of the ham sandwich theorem. We prove among other results, the following implications

$$B(n-1,k-1) \leftarrow B(n,k) \leftarrow C(n,k) \rightarrow K(n,k) \rightarrow K(n-1,k-1)$$

where $K(n,k)$ is an important instance of the Knaster's conjecture so that $K(n,n-1)$ reduces to the Borsuk–Ulam theorem, $B(n,k)$ is a R. Rado type statement about $(k+1)$ measures in R^n where $B(n,n-1)$ turns out to be the ham sandwich theorem and $C(n,k)$ is a topological statement, established in this paper in the case $C(n,n-2)$, $n=3$ or $n \geq 5$.

Introduction

This paper can be seen as a sequel to our paper [10] which was written, among other reasons, to give evidence that the well known "Ham sandwich theorem" can be seen as a distant relative of Helly's convexity theorem. More precisely it was shown in [10], see the Theorem $A(n,k)$ below, that both the ham sandwich theorem and R. Rado's theorem on the general measure [9], which is known to be a measure theoretic equivalent of Helly's convexity theorem, belong to the same

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family of results about extremal properties of measurable sets or measures in the n-dimensional Euclidean space R^n. The papers [1] and [3] are recommended as good overviews of some recent applications of Borsuk–Ulam type results in Combinatorics and Combinatorial Geometry. Our initial intention was to prove another extremal result, see the statement $B(n,k)$ below, about measures defined in R^n, which also contains the ham sandwich theorem as a special case and has a clear geometric and combinatorial meaning. Soon after that, it turned out that an important instance of the well known Knaster's conjecture (denoted by $K(n,k)$ in §3 is a "relative" of $B(n,k)$ in the sense that they both follow from the same topological conjecture $C(n,k)$. We prove here only the case $C(n,n-2)$, $n=3$ or $n \geq 5$ (see Theorem 2.2), but already this result itself has interesting consequences. For the reader's convenience, we include the following graph of implications which are, together with some other results, established in this paper.

$$C(n,k)$$

$$\swarrow \qquad \searrow$$

$$B(n-1,k-1) \leftarrow B(n,k) \qquad K(n,k) \rightarrow K(n-1,k-1)$$

Note that the horizontal implications are included because this paper contains only a proof of the particular cases of $C(n,k)$ mentioned above but we are convinced that $C(n,k)$ holds in general and believe that the proof should not be much more complicated than the proof of Theorem 2.2. Let us also remark that $C(n,n-1)$ is true, $B(n,n-1)$ is the ham sandwich theorem and $K(n,n-1)$ is the Borsuk–Ulam theorem.

THEOREM $A(n,k)$ ([10]): *Let $\mu_0,\mu_1,\dots,\mu_k$, $0 \leq k \leq n-1$, be a collection of σ-additive probability measures defined on the σ-algebra of all Borel sets in R^n. Then there exists a k-dimensional affine subspace $d \subseteq R^n$ such that for every closed halfspace $H(v,\alpha) := \{x \in R^n \mid < x,v > \leq \alpha\}$ and every $i \in \{1,2,\dots,k\}$, $d \subseteq H(v,\alpha)$ implies $\mu_i(H(v,\alpha)) \geq 1/(n-k+1)$.*

The proof of this theorem was based on a result about nonexistence of a nowhere zero cross-section of a vector bundle over a Grassmannian, which was proved with the aid of the theory of Stiefel–Whitney characteristic classes. Let us formulate now the statement $B(n,k)$ which just like $A(n,k)$ contains the ham sandwich theorem as a special case. Roughly speaking, $B(n,k)$ says that, given

$(k+1)$ measures in R^n, one can dissect R^n into wedge-like cones which all have the same measure from the point of view of each of given measures. The analogy of $B(n,k)$ with $A(n,k)$ goes further because like in the case of $A(n,k)$, a natural vector bundle over the Grassmannian arises and it suffices to prove that this bundle does not admit a nowhere zero continuous cross-section. However, it seems that this result, called $C(n,k)$, is more difficult to establish and as evidence for this we show in §3 that one of the main cases of the well known Knaster's conjecture can be reduced to exactly the same topological question. Nevertheless, we are able to establish $B(n,k)$ in some particular cases and the isolation of a clear topological result responsible, should hopefully lead to the complete solution. We intend and hope to return to this topological question in the future. Before we formulate $B(n,k)$ let us fix some notation. Let $\Delta = \operatorname{conv}\{a_0,\dots,a_m\} \subseteq R^n$, $a = 1/(m+1)(a_0+\dots+a_m)$ is the barycenter of Δ and Δ_i the face of Δ opposite to the vertex a_i. Let $\operatorname{cone}(a,\Delta_i) := \bigcup_{\lambda\geq 0}(a+\lambda(\Delta_i - a))$, $\operatorname{cone}(\Delta_i) :=$ cone $(0,\Delta_i)$. The following equality $p :=$ $\operatorname{aff}(\Delta) = \bigcup_{0\leq i\leq m}$ $\operatorname{cone}(a,\Delta_i)$ is obvious and if $p^\perp$ is the linear subspace of R^n orthogonal to p and $D_i := p^\perp +$ $\operatorname{cone}(a,\Delta_i)$ a wedge-like cone in R^n then $R^n = \bigcup_{0\leq i\leq m} D_i$ and $\operatorname{int}(D_i)\bigcap\operatorname{int}(D_j) = \emptyset$ for $i \neq j$.

Given a m-dimensional regular simplex $\Delta \subseteq R^n$, the associated m-dimensional regular simplicial dissection $\mathcal{D}(\Delta) := \{D_i\}_{i=0}^m$ will be simply called the dissection associated with Δ. Let us note that for $m=1$ we obtain a hyperplane dissection of R^n.

In order to avoid unnecessary complications which may arise in dealing with general probabilities, we will restrict ourselves to the class P of "admissible" probability Borel measures, that is the probability measures which are weak limits (see [2]) of probability measures absolutely continuous with respect to the Lebesgue measure. This permits us to assume in all arguments that the measures in question are continuous and then pass to the limit. Note that all combinatorially interesting measures, including the measures supported by finite sets in R^n, belong to P.

CONJECTURE $B(n,k)$: *Let $\mu_0,\mu_1,\dots,\mu_k$ be a family of admissible, probability measures, $0\leq k\leq n-1$, where admissible means that all measures belong to the class P defined above. Then there exists a $(n-k)$-dimensional dissection $\mathcal{D}(\Delta) := \{D_i\}_{i=0}^{n-k}$ of R^n such that for every i, $0\leq i\leq k$, and s, $0\leq s\leq n-k$, $\mu_i(D_s)\geq 1/(n-k+1)$. Specially, if all measures are continuous then $\mu_i(D_s) = \mu_i(D_t)$ for all i, $0\leq i\leq k$, and s,t, $0\leq s,t\leq n-k$.*

1. $C(n,k)$

$C(n,k)$ is an abbreviation for the main topological statement mentioned in the introduction. Before it is formulated, let us recall some familiar facts about vector bundles. All groups and base spaces of vector bundles are assumed to be compact. Let $P \to X$ be a principal G-bundle over X, V a left G-vector space and $P \times_G V$ the associated vector bundle with fiber V. If H is a closed subgroup of G then P is a principal H-bundle over P/H and $P \times_H V$ is identified with the pull back of the bundle $P \times_G V$ along the map $P/H \to P/G = X$. A canonical example is $P = V_{n,k}$, the Stiefel manifold, as a principal $O(k)$-bundle over $X = G_{n,k}$ the Grassmann manifold. If $H = T^k \cong Z_2 \oplus \cdots \oplus Z_2$, k copies of Z_2, is the "maximal discrete" torus in $O(k)$ then $P/H = \widetilde{V}_{n,k}$ is the flag manifold and both $V_{n,k} \times_{O(k)} R^k$ and $V_{n,k} \times_{T^k} R^k$ are canonical k-plane bundles over $G_{n,k}$ and $\widetilde{V}_{n,k}$ respectively. The nontriviality of the canonical bundle over $\widetilde{V}_{n,k}$ from the point of view of existence of cross-sections of associated Whitney sums was analyzed in [5] and this topological result played a key role in the proof of the Theorem $A(n,k)$ stated above. For a similar reason, we will primarily be interested in an analogously constructed bundle in which the group T^k is replaced by the cyclic group $H = Z_{k+1}$ which is seen in $O(k)$ as the group of all isometries determined as cyclic permutations of vertices of a fixed regular simplex centered at the origin in R^n. The analog of the flag bundle is the bundle $V^*_{n,k} := V_{n,k}/Z_{k+1}$ which will be, till the end of this paper, called the star bundle. As before $\xi := V_{n,k} \times_{Z_{k+1}} R^k$ is the k-plane bundle over $V^*_{n,k}$ which is the pull back of the canonical bundle over $G_{n,k}$ along the projection $V^*_{n,k} \to G_{n,k}$. Now we are ready to formulate the "Hauptvermutung" of this paper.

$C(n,k)$: *The Whitney sum $\xi^{\oplus k}$ of k copies of the canonical vector bundle over $V^*_{n,n-k}$ does not admit a nowhere zero continuous cross-section.*

PROPOSITION 1.1: $C(n,k) \Rightarrow B(n,k)$.

Before we prove this implication, we need an auxiliary proposition which is just $B(n,0)$ slightly extended for our purposes.

PROPOSITION 1.2: *Let μ be a continuous probability measure on R^n. Let $\triangle$ be a fixed regular n-dimensional simplex in R^n centered at the origin, $x + \triangle$ its translate and $\mathcal{D}(x + \triangle)$ the associated dissection of R^n defined above. Then $\mathcal{O}(\mu, \triangle) := \{x \in R^n | (\forall L \in \mathcal{D}(x+\triangle))\ \mu(L) \geq 1/(n+1)\}$ is a nonempty compact, convex set.*

Proof: Let us prove first that $\mathcal{O}(\mu, \Delta)$ is nonempty. We start with the assumption that $\Delta :=$ conv$\{e_0, e_1, \dots, e_n\}$ is chosen very big so that the following holds. Let $\nabla :=$ conv$\{\hat{e}_0, \hat{e}_1, \dots, \hat{e}_n\}$ where $\hat{e}_i := 1/n \sum_{j \neq i} e_j$ is the barycenter of the face Δ_i opposite to the vertex e_i and Δ' the simplex obtained from ∇ by the same procedure, hence $\Delta' = (1/n^2)\Delta$. The assumption that Δ is big means that $\mu(\text{int}(\Delta')) > n/(n+1)$. If $D_i = \text{cone}(\Delta_i) := \bigcup_{\lambda \geq 0} \lambda \Delta_i$ and $x \in \Delta$, let $\varphi(x) :=$ conv$\{\hat{e}_i | \mu(x + D_i) \geq 1/(n+1)\}$. Now, φ is easily seen to be a multivalued, convex, compact, uppersemicontinuous function $\varphi \colon \Delta \to \Delta$. By the Kakutani's fixed point theorem for some $x \in \Delta$, $x \in \varphi(x)$ and it remains to be checked that this can happen only if $\varphi(x) = \nabla$. Otherwise, x belongs to a proper face of ∇, say $x \in \varphi(x) =$ conv$\{\hat{e}_i | i \in I\} \neq \nabla$. If $x = \sum \alpha_j \hat{e}_j$, $\alpha_j = 0$ for $j \notin I$, and $\alpha_v := \max\{\alpha_j\}_{j \in I}$ then

$$(1) \qquad (x + D_v) \bigcap \text{int}(\Delta') = \emptyset,$$

hence $\mu(x + D_v) < 1/(n+1)$ which contradicts $v \in I$. In order to give some firm algebraic evidence for (1) let $y = \sum \lambda_j e_j \in D_v$ where $\lambda_j \geq 0$ and $\lambda_v = 0$. From $x = \sum_{j \in I} \alpha_j \hat{e}_j = \sum_{j \in I} \alpha_j \sum_{i \neq j} (1/n) e_i = \sum_i (\sum_{j \in I \setminus \{i\}} \alpha_j / n) e_i$ follows $x + y = \sum (\xi_i + \lambda_i) e_i$ where $\xi_i = \sum_{j \in I \setminus \{i\}} \alpha_j / n$. In order to show that $x + y \notin \text{int}(\Delta')$ let us find the barycentric coordinates β_j of $x + y$ relative to Δ'. Easy calculation shows that $\beta_j = n^2(\xi_j + \lambda_j) - (n-1) - n^2/(n+1)\lambda$, where $\lambda = \sum_{i=0}^n \lambda_i$, in particular $\beta_v = n^2 \xi_v - (n-1) - n^2/(n+1)\lambda$. From the definition of ξ_v and maximality of α_v follows $\xi_v \leq (n-1)/n^2$ so $\beta_v \leq -n^2/(n+1)\lambda \leq 0$. Hence $x + y \notin \text{int}(\Delta')$.

Let us show now that $\mathcal{O}(\mu, \Delta)$ is a convex set. Without loss of generality it is enough to prove that $0, x \in \mathcal{O}(\mu, \Delta)$ imply $tx \in \mathcal{O}(\mu, \Delta)$ for $0 \leq t \leq 1$. Let $D_i^t := tx + D_i$. We are supposed to show that $(\forall i)\mu(D_i) = \mu(D_i^1) = 1/(n+1)$ imply $(\forall i)\mu(D_i^t) = 1/(n+1)$. Let $x = \sum \lambda_i e_i$, $\sum \lambda_i = 1$. It is a trivial but useful fact that $x \in D_v$ iff $\lambda_v = \min\{\lambda_j\}_{j=0}^n$ so for convenience assume $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n$.

Claim: $(\forall t)(\forall k) D_0^t \cup \cdots \cup D_k^t \subseteq D_0 \cup D_1 \cup \cdots \cup D_k$.

Actually, it is enough to show $D_k^1 \subseteq D_0 \cup D_1 \cup \cdots \cup D_k$. Let $x + y \in D_k^1 := x + D_k$ where $y = \sum_{i=0}^n \mu_i e_i$, $\mu_i \geq 0$, $\mu_k = 0$. Obviously, $x + y = \sum_{i=0}^n (\lambda_i + \mu_i) e_i \in \bigcup_{j=0}^n D_j$. Since $\lambda_k = \lambda_k + \mu_k \leq \min\{\lambda_j + \mu_j | j \leq k\}$ we observe that

$$\lambda_\nu + \mu_\nu = \min\{\lambda_j + \mu_j | 0 \leq j \leq n\}$$

implies $\nu \le k$, i.e. $x + u \in D_\nu \subseteq D_0 \cup \cdots \cup D_k$.

From the claim one deduces that for every k

$$D_0^1 \cup \cdots \cup D_k^1 \subseteq D_0^t \cup \cdots \cup D_k^t \subseteq D_0 \cup \cdots \cup D_k.$$

From here and the continuity of the measure μ one has

$$\mu(D_0^1) + \cdots + \mu(D_k^1) \le \mu(D_0^t) + \cdots + \mu(D_k^t) \le \mu(D_0) + \cdots + \mu(D_k)$$

and the convexity of $\mathcal{O}(\mu, \Delta)$ is an immediate consequence.

Remark: Let us note that the Proposition 1.2 in general does not hold if μ is assumed to be an arbitrary admissible measure. Indeed, if μ is the measure defined in R^2 such that $\mu(\{a\}) = \mu(\{b\}) = 1/2$ where $a \ne b$, then for a generic simplex Δ, $\mathcal{O}(\mu, \Delta)$ consists of the union of two noncollinear segments. ∎

Proof of Proposition 1.1: Let us establish the implication $C(n,k) \Rightarrow B(n,k)$ first for a sequence $\mu_0, \mu_1, \ldots, \mu_k$ of continuous probability measures. We identify the Stiefel manifold $V_{n,n-k}$ with the space of all sequences $(a_0, a_1, \ldots, a_{n-k}) \in (R^n)^{n-k+1}$ (ordered simplexes) where a_i, $0 \le i \le n-k$, are vertices of a regular $(n-k)$-dimensional simplex in R^n centered at the origin. For each of the measures μ_i, $\mathcal{O}(\mu_i, \Delta)$ can be seen as a multivalued, convex, compact cross-section of the $(n-k)$-plane bundle γ over $V_{n,n-k}$ which is the pull back of the canonical bundle over the Grassmannian $G_{n,n-k}$ along the projection $V_{n,n-k} \to G_{n,n-k}$. The obvious invariance of the set $\mathcal{O}(\mu_i, \Delta)$ under the action of the cyclic group of isometries which permutes elements of $(a_0, a_1, \ldots, a_{n-k}) \in V_{n,n-k}$ permits us to see $\mathcal{O}(\mu_i, \Delta^*) := \mathcal{O}(\mu_i, \Delta)$ as a multivalued cross-section of the canonical $(n-k)$-plane bundle ξ over the star manifold $V_{n,n-k}^*$ where $\Delta^* \in V_{n,n-k}^*$ is the orbit of Δ. Then, $\overline{\varphi}(\Delta^*) := \mathcal{O}(\mu_0, \Delta^*) \oplus \cdots \oplus \mathcal{O}(\mu_k, \Delta^*)$ is seen as a multivalued section of the Whitney sum $\xi^{\oplus(k+1)}$ of $(k+1)$ copies of the canonical bundle. In case $B(n,k)$ is false, i.e. if $\bigcap_{i=0}^k \mathcal{O}(\mu_i, \Delta^*) = \emptyset$ for all $\Delta^* \in V_{n,n-k}^*$ we observe that $\overline{\varphi}(\Delta^*) \cap \mathrm{Diag} = \emptyset$ where Diag is the diagonal subbundle of $\xi^{\oplus(k+1)}$. Let $L\colon \xi^{\oplus(k+1)} \to \xi^{\oplus k}$ be a morphism of vector bundles which is on each fibre defined by $L(V_0, \ldots, V_k) := (V_1 - V_0, \ldots, V_k - V_0)$. Then $\varphi := \hat{L} \circ \hat{\varphi}$ is a nowhere zero, multivalued, compact, convex section of $\xi^{\oplus k}$. It was proved in [10] (Proposition 1) that under these conditions there exists a nowhere zero, single-valued, continuous cross-section of the bundle $\xi^{\oplus k}$ which contradicts $C(n,k)$. Hence, $B(n,k)$ holds if all measures are continuous.

Now, if $\mu_0, \dots, \mu_k$ are measures which are weak limits $P_m^i \Rightarrow \mu_i$, $m \to +\infty$, of continuous probability measures, an easy compactness argument permits us to assume that there exists a sequence of simplexes Δ^m converging to a regular simplex Δ such that Δ^m is obtained from $B(n,k)$ applied on the family P_m^i, $0 \le i \le k$. To be completely precise, we assume here and further on that all measures μ_i have bounded support and it is easily seen by the same argument that this is not a loss of generality. Let us convince ourselves that the dissection $\mathcal{D}(\Delta)$ proves $B(n,k)$ for measures μ_i. Let $D_i \in \mathcal{D}(\Delta)$ and let $D_i^m \in \mathcal{D}(\Delta^m)$ be the corresponding wedge-like cones which converge to D_i. Also, let supports of all measures be contained in a ball O and let $U \supseteq D_i$ be an open set. Then, be the well known characterization of weak convergence (see [2], T.2.1.) $\limsup_m P_m^j(U) \le \mu_j(U)$. On the other hand, $O \cap D_i^m \subseteq U$ if m is big enough. Hence, $\mu_j(U) \ge 1/(n-k+1)$ for every U and by the σ-additivity of $\mu_j, \mu_j(D_i) \ge 1/(n-k+1)$ for all $0 \le i, j \le n-k$.

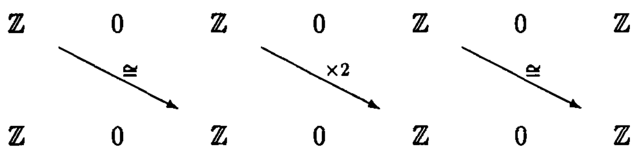
2. Proof of $B(n, n-2)$

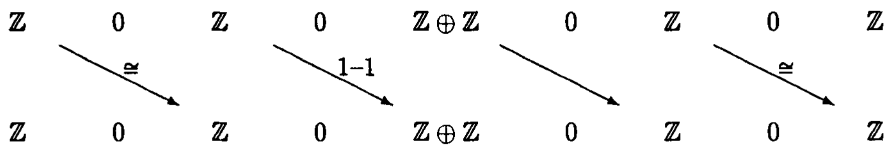
PROPOSITION 2.1: $B(n,k) \Rightarrow B(n-1,k-1)$

Proof: If suffices to check $B(n-1,k-1)$ for continuous measures μ_i, with bounded support, $\mu_i(B) = 1$, $1 \le i \le k$, where $B = \{x \in R^n \mid \| x \| \le r\}$, $r > 0$. Let $\overline{\mu}_0^m$, $m \in N$, be the measure concentrated at the point $z_m = (0, R_m) \in R^{n-1} \times R = R^n$, $R_m \to +\infty$. By $B(n,k)$ applied on the measures $\overline{\mu}_0^m, \overline{\mu}_1, \dots, \overline{\mu}_k$, where $\overline{\mu}_j(A) := \mu_j(A \cap R^{n-1})$, $1 \le j \le k$, there exists a dissection $\mathcal{D}_n(\Delta_m) = \{D_i^m\}_{i=0}^{n-k}$, $p_m :=$ aff(Δ_m), $\Delta_m =$ conv$\{a_i^m\}_{i=0}^{n-k}$, $a_m = 1/n(a_0^m + \cdots + a_{n-k}^m)$, such that $\overline{\mu}_i(D_v^m) \ge 1/(n-k+1)$ for all $0 \le i \le k$. From the definition of μ_0^m there follows $z_m \in a_m + p_m^\perp$ and if $R_m \to +\infty$, Δ_m becomes more and more horizontal. By compactness we can assume that $\Delta_m \to \Delta$ where $\Delta \subseteq R^{n-1}$ and the associated dissection $\mathcal{D}(\Delta)$ proves $B(n-1,k-1)$. ∎

THEOREM 2.2: *$C(n, n-2)$ is true for $n = 3$ and all $n \ge 5$.*

Proof: We are to show that $\xi^{\oplus(n-2)}$ does not admit a nowhere zero continuous cross-section. Let $G_{n,k}^+$ denote the Grassmann manifold of all oriented k-planes in R^n. Since Z_3 acts on V_{n-2} as a subgroup of SO(2) $\cong S^1$, the bundle ξ is seen as the pull back of the canonical, oriented, 2-dimensional bundle ζ over $G_{n,2}^+$. Hence, ξ itself is oriented and the Euler class $x = e(\xi)$ is well defined. Nonvanishing of the Euler class is one of sufficient conditions for nonexistence of a continuous, nonzero cross section, hence it is enough to show that $x^{n-2} = e(\xi^{\oplus(n-2)}) \neq 0$. First of all, let us compute the cohomology of $G^+_{n,2}$ with integer coefficients. We will assume from now on that $n \geq 5$ and leave it to the reader to convince himself that a similar proof holds also for $n = 3$. The E_2-term of the spectral sequence associated to the fibration $S^1 \to V_{n,2} \to G^+_{n,2}$ is easily reconstructed from the fact that $H^*(V_{n,2}, Z) \cong H^*(S^{n-1} \times S^{n-2}; Z)$ for n even, and for n odd all groups are zero except for the groups $H^0(V_{n,2}, Z) \cong H^{2n-3}(V_{n,2}, Z) \cong Z$ and $H^{n-1}(V_{n,2}, Z) \cong Z_2$. For n odd, one deduces that $H^{2i}(G^+_{n,2}, Z) \cong Z$, $0 \leq i \leq n-2$, and for all i, $H^{2i+1}(G^+_{n,2}, Z) \cong 0$. For n even the answer is similar except in the middle dimension when $H^{n-2}(G^+_{n,2}, Z) \cong Z \oplus Z$. For n odd, all differentials are isomorphisms except d_2: $E_2^{n-3,1} \to E_2^{n-1,0}$ which is multiplication by 2. For n even the answer is similar except for the case of the middle differential d_2: $E_2^{n-2,1} \to E_2^{n,0}$ but we will not need this information here. The diagrams below show the E_2-term for the case of $G^+_{5,2}$ and $G^+_{6,2}$ respectively.





A generator of $H^2(G^+_{n,2}, \mathbb{Z})$ is identified as the Euler class $x = e(\zeta)$ of ζ since $x = d_2(y)$ where y is the generator of $E_2^{0,1}$ coming from the orientation of ζ. Let us assume that n is odd. By using the multiplicative structure in $E_2^{*,*}$, one obtains that x^j is a generator of $E_2^{2j,0}$ and $x^j y$ is a generator of $E_2^{2j,1}$ for $0 \leq j \leq (n-3)/2$. For $(n-1)/2 \leq j \leq n-2$, $x^j = 2g$ where $g \in E_2^{2j,0}$ is a generator. In a special case which we are particularly interested in, one has $x^{n-2} = 2g$. Since $G^+_{2m,2}$ can be put in a "sandwich" between two odd Grassmannians, $G^+_{2m-1,2} \to G^+_{2m,2} \to G^+_{2m+1,2}$, the naturality of the Euler class implies ($m \neq 2$) that x^{2m-3} is also a multiple by two of a generator in $H^{2(2m-3)}(G^+_{2m,2})$. Since d_2: $E_2^{2m-4,1} \to E_2^{2m-2,0}$ is an isomorphism and $d_2(x^{2m-3}y) = x^{2m-2}$, we conclude that for all $n \geq 5$,

$x^{n-2} = 2g$ for a generator $G \in E_2^{n-2,0} \cong H^{2(n-2)}(G^+_{n,2}, Z) \cong Z$. Let us show now that $e(\xi^{\oplus(n-2)}) \in H^{2(n-2)}(V^*_{n,2}, Z)$ is also different from zero. From the commutative diagram of fibrations

$$\begin{array}{ccccc} S^1 & \longrightarrow & V_{n,2} & \longrightarrow & G^+_{n,2} \\ {\scriptstyle \times 3}\downarrow & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & V^*_{n,2} & \xrightarrow{p} & G^+_{n,2} \end{array}$$

and the naturality of E_2-terms of the corresponding spectral sequences, one obtains among other things that d_2: $E_2^{n-4,1} \to E_2^{n-2,0}$, in the spectral sequence associated to $V^*_{n,2}$, is multiplication by 3. This means that $x^{n-2} = 2g \in E_2^{n-2,0}$ survives to ∞, so $e(\xi^{\oplus(n-2)}) = p^*(x^{n-2}) \neq 0$ which proves that $\xi^{\oplus(n-2)}$ does not admit a continuous, nonzero cross section. ∎

COROLLARY 2.3: *$B(n, n-2)$ is true. In other words if $\mu_1, \ldots, \mu_{n-1}$ is a family of admissible, probability measures defined in R^n, then there exists a 2-dimensional dissection $\mathcal{D}(\Delta) := \{D_0, D_1, D_2\}$ of R^n such that for every i, $0 \le i \le 2$, and s, $1 \le s \le n-1$, $\mu_i(D_s) \ge 1/3$.*

Proof: Indeed, $B(n, n-2)$, $n \ge 5$, holds by Proposition 1.1 and the theorem above, so $B(n, n-2)$ follows in the few remaining cases from Proposition 2.1. ∎

3. Knaster's Conjecture

Probably the most natural extension of the Borsuk–Ulam theorem is the following conjecture which in this or similar form is well known as the Knaster's conjecture (see [3] [6]).

KNASTER'S CONJECTURE: *Let f: $S^{n-1} \to R^k$ be a continuous map and $A = \{a_0, \ldots, a_{n-k}\} \subseteq S^{n-1}$ a finite set of points. Then there exists an isometry $o \in O(n)$ such that $f(o(a_i)) = f(o(a_j))$ for all i, j.*

One can ask for the largest m such that the statement above still holds where m is the cardinality of the set A. This question, or more precisely the question for which n, k and $A \subseteq S^{n-1}$ the conjecture above holds will be referred to as the Knaster's problem.

It is a nice observation of V. V. Makeev, [7], that if $d := \dim\ \mathrm{aff}(A)$ a necessary condition for a positive solution to the Knaster's problem is the following inequality

$$d(2n - d - 1) > 2k(m - 1), \tag{1}$$

so in case all points in A are in general position one has $m < 2(n - k)$.

The main objective of this paragraph is to show that one of the main instances of the Knaster's conjecture follows from $C(n, k)$. As a consequence, we are able to prove the conjecture in those few cases for which $C(n, k)$ is known.

Let us start with the following well known observation which provides an important link between equivariant maps and cross-sections of vector bundles.

PROPOSITION 3.1: *Let P be a principal G-bundle over X and V a left G-vector space. Then all continuous cross-sections of the induced vector bundle $P \times_G V$ are in $1 - 1$ correspondence with the G-equivariant maps $f\colon P \to V$ where both P and V are seen as right G-spaces, i.e., for $g \in G$ and $v \in V$, $(v)g := g^{-1}v$.*

Let us return now to the inequality (1) above. By inspecting th left hand side of (1) one observes that the likelihood for a counterexample to exist gets bigger if dim aff(A) gets smaller. Roughly speaking, in this case $o(A)$ is determined by very few members of A for every $o \in O(n)$, which doesn't give much chance for a generic map f to fulfill the coincidence condition above for some $o \in O(n)$. So, it seems natural to pay special attention to the case $\dim \mathrm{aff}(A) = n - k$. Specially, the simplest situation occurs if A is the set of vertices of a regular $(n - k)$-dimensional simplex in which case we denote the Knaster's conjecture above by $K(n, k)$.

PROPOSITION 3.2: $C(n, k) \Rightarrow K(n, k)$.

Proof: As before, the Stiefel manifold is identified with the manifold

$$V_{n,m} := \{(a_0, \ldots, a_m) |\ a_i \in S^{n-1}, \mathrm{conv}\{a_i\}_{i=0}^m \text{ is a regular simplex}\}.$$

$V_{n,m}$ is, as before, seen as the principal Z_{m+1}-bundle over the star manifold $V^*_{n,m} := V_{n,m}/Z_{m+1}$. For every $f\colon S^{n-1} \to R^k$ one defines $F\colon V_{n,n-k} \to W_{n,k}$, where $W_{n,k}$ is the orthogonal complement to the diagonal $d \cong R^k$ in $R^k \times \cdots \times R^k \cong R^{k(n-k+1)}$, by $F(a_0, a_1, \ldots, a_{n-k}) := \Pr(f(a_0), \ldots, f(a_{n-k}))$ where $\Pr\colon R^k \times \cdots \times R^k \to W_{n,k}$ is the orthogonal projection. Now, if $K(n, k)$ were false then there would exist a nowhere zero Z_{n-k+1}-equivariant map $F\colon V_{n,n-k} \to W_{n,k}$. By Proposition 3.1 this is equivalent to existence of a nonzero continuous cross-section of the bundle $V_{n,n-k} \times_{Z_{n-k+1}} W_{n,k}$. This contradicts $C(n,k)$ because the last bundle is easily seen to be isomorphic to the bundle $\xi^{\oplus k}$ where ξ is the canonical vector bundle over $V^*_{n,n-k}$. ∎

So both $K(n,k)$ and $B(n,k)$ are consequences of $C(n,k)$. Let us show now that the formal similarity between statements $K(n,k)$ and $B(n,k)$ goes further by showing that $K(n,k)$ satisfies a proposition similar to Proposition 2.1.

PROPOSITION 3.3: $K(n,k) \Rightarrow K(n-1,k-1)$.

Proof: Let $f\colon S^{n-2} \to R^{k-1}$ be a continuous map. Let $S^{n-1} \cong S^{n-2} * S^0$, that is each point $x \in S^{n-1} \subseteq R^n$ is represented uniquely as $x = \cos(t)e + \sin(t)y$, where $e \in \{e_n, -e_n\} = S^0$, $y \in S^{n-2} \subseteq R^{n-1}$ and $0 \le t \le \pi/2$. Let $\hat{f}\colon S^{n-1} \to R^k$ be defined by the formula $\hat{f}(\cos(t)e + \sin(t)y) := \cos(t)e + \sin(t)f(y)$. By $K(n,k)$ there exists a regular simplex $\Delta = \{a_0, \ldots, a_{n,k}\} \subseteq S^{n-1}$ such that $\hat{f}(a_i) = \hat{f}(a_j)$ for all i,j. By the definition of $\hat{f}$ it is clear that this complex must be horizontal, i.e. Δ can be taken in S^{n-2}. Hence, $f(a_i) = f(a_j)$ for all i,j and $K(n-1,k-1)$ holds. ∎

From Theorem 2.2 and Propositions 3.2 and 3.3 one immediately obtains the following corollary.

COROLLARY 3.3: *$K(n,n-3)$ is true. In other words, for every continuous map $f\colon S^{n-1} \to R^{n-2}$ there exist points a_0, a_1, a_2 which are vertices of a regular triangle which is inscribed in a great circle of S^{n-1} such that $f(a_i) = f(a_j)$ for all i,j.*

Proof: From Theorem 2.2 and Proposition 3.2 one obtains $K(n,n-2)$ for $n \ge 5$, so the corollary follows from Proposition 3.3. ∎

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